Recent progress on nonabelian tensor squares of groups

Russell D. Blyth

Saint Louis University

Preliminary report on joint work with
Francesco Fumagalli (Firenze)
and Marta Morigi (Bologna)

Special Session on Computational Group Theory
2008 Fall A.M.S. Central Section Meeting · October 18, 2008

Definition of the nonabelian tensor square

Definition

The nonabelian tensor square $G \otimes G$ of the group G is the group generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$$

for all $g, g, h, h' \in G$, where ${}^h g = hgh^{-1}$ is conjugation on the left.

For consistency, we define the commutator $[h, g]$ of group elements h and g to be $hgh^{-1}g^{-1}$.

Definition of the nonabelian exterior square

Let $\nabla(G) = \langle g \otimes g \mid g \in G \rangle \leq G \otimes G$.

$\nabla(G) \leq Z(G \otimes G)$.

Definition

The *nonabelian exterior square* $G \wedge G$ of the group G is the factor group $G \otimes G / \nabla(G)$

Commutative diagram from Brown and Loday (1987)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{\mathsf{ab}}) & \stackrel{\psi}{\longrightarrow} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

where all the sequences are exact and the short exact sequences are central. $\Gamma(G^{\mathsf{ab}})$ is the Whitehead quadratic functor.

Results of Brown, Johnson and Robertson (1987)

Proposition

If G is a free group, then $G \otimes G \cong G' \times \Gamma(G^{\mathsf{ab}})$.

If G is free of finite rank $n \geq 2$, then G' is free of countably infinite rank and $\Gamma(G^{\mathsf{ab}})$ is free abelian of rank $\frac{n(n+1)}{2}$.

Definition

A *covering group* $\hat{G}$ of a group G is a central extension

$$1 \longrightarrow H_2(G) \xrightarrow{\iota} \hat{G} \longrightarrow G \longrightarrow 1,$$

where Im $\iota \subseteq \hat{G}'$.

Proposition

If $\hat{G}$ is a covering group of G, then there is a map $\eta : G \wedge G \to \hat{G}'$, which is an isomorphism if $H_2(G)$ is finitely generated.

The group $\nu(G)$

Definition (Ellis and Leonard (1995), Rocco (1991))

Let G be a group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and let G^φ be an isomorphic copy of G via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. Define the group $\nu(G)$ to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^\varphi | \mathcal{R}, \mathcal{R}^\varphi, {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi], \forall x, g, h \in G \rangle.$$

The groups G and G^φ embed isomorphically into $\nu(G)$. By convention the labels G and G^φ also denote their natural isomorphic copies in $\nu(G)$.

The group $\nu(G)$

Theorem (Ellis and Leonard (1995), Rocco (1991))

Let G be a group. The map

$$\phi : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$$

defined by $\phi(g \otimes h) = [g, h^{\varphi}]$ for all g and h in G is an isomorphism.

Note that $\nu(G)$ has $2|\mathcal{G}|$ generators, a significant reduction from the number of generators of $G \otimes G$. Ellis and Leonard show that the number of relations for $\nu(G)$ can be pruned to a degree that depends on the size and structure of the center of G.

Compute $G \otimes G$ by computing a small finite presentation for $\nu(G)$ and using it to determine its subgroup $[G, G^{\varphi}]$.

Properties of $\nu(G)$

Theorem (Rocco; Ellis and Leonard)

Let G be a group.

(i) *If G is finite then $\nu(G)$ is finite.*

(ii) *If G is a finite p-group then $\nu(G)$ is a finite p-group.*

(iii) *If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c+1$.*

(iv) *If G is solvable of derived length d then $\nu(G)$ is solvable of derived length at most $d+1$.*

(v) *Let $\iota : [G, G^{\varphi}] \to \nu(G)$ be the natural inclusion map and let $\xi : \nu(G) \to G \times G$ be the homomorphic extension of the map sending the generator $g \in G$ of $\nu(G)$ to $(g, 1)$ and the generator $g^{\varphi} \in G^{\varphi}$ of $\nu(G)$ to $(1, g)$. Then*

$$1 \longrightarrow [G, G^{\varphi}] \xrightarrow{\iota} \nu(G) \xrightarrow{\xi} G \times G \longrightarrow 1$$

is a short exact sequence.

Properties of $\nu(G)$

Lemma (Rocco; B, Moravec, and Morse)

Let G be a group. The following relations hold in $\nu(G)$:

(i) ${}^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *and* ${}^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *for all* g_1, g_2, g_3, g_4 *in* G;

(ii) $[g_1^{\varphi},g_2,g_3] = [g_1,g_2,g_3^{\varphi}] = [g_1^{\varphi},g_2,g_3^{\varphi}] = [g_1,g_2^{\varphi},g_3] = [g_1^{\varphi},g_2^{\varphi},g_3] = [g_1,g_2^{\varphi},g_3^{\varphi}]$ *for all* g_1, g_2, g_3 *in* G;

(iii) $[g_1,[g_2,g_3]^{\varphi}] = [g_2,g_3,g_1^{\varphi}]^{-1}$;

(iv) $[g,g^{\varphi}]$ *is central in* $\nu(G)$ *for all* g *in* G;

(v) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* g_1, g_2 *in* G;

(vi) $[g,g^{\varphi}] = 1$ *for all* g *in* G'.

Polycyclic groups

Theorem (B, Moravec, and Morse)

Let G be a polycyclic group with a finite presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and polycyclic generating set $\mathfrak{G}$. Then

(i) *The nonabelian tensor square $G \otimes G$ is polycyclic.*

(ii) *The group $\nu(G)$ is polycyclic.*

(iii) *The group $\nu(G)$ has a finite presentation that depends only on $\mathcal{G}$, $\mathcal{R}$ and $\mathfrak{G}$.*

(iv) *The nonabelian tensor square $G \otimes G$ is generated by the set*

$$\{\mathfrak{g}^{\pm 1} \otimes \mathfrak{h}^{\pm 1} \mid \ for\ all\ \mathfrak{g}, \mathfrak{h}\ in\ \mathfrak{G}\}.$$

These results support hand and computer calculations, for example, using a polycyclic quotient algorithm.

Definition of $\mathcal{L}_G$

Definition

Let G be a group and let

$$G = G_n \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1$$

be a subnormal series for G. Let $\mathcal{T}_i$ denote a transversal for G_{i-1} in G_i and let $\mathcal{G}_i$ denote a lift of a generating set for G_i/G_{i-1} to $\mathcal{T}_i$. Set

$$\mathcal{L}_i = \begin{cases} \mathcal{G}_i & \text{if } G_i/G_{i-1} \text{ is abelian} \\ \mathcal{T}_i & \text{otherwise.} \end{cases}$$

Then the set $\mathcal{L}_G$ relative to the subnormal series $G = G_n \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1$ is defined as

$$\mathcal{L}_G = \cup_{i=1}^n \mathcal{L}_i.$$

Theorem (B, Moravec, and Morse)

Let G be a group with presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and let $\mathcal{S}$ be any subnormal series of G. Then $\nu(G)$ is given by the following presentation:

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} \mid \mathcal{R}, \mathcal{R}^{\varphi}, {}^{x}[a, b^{\varphi}] = [{}^{x}a, ({}^{x}b)^{\varphi}], {}^{x^{\varphi}}[a, b^{\varphi}] = [{}^{x}a, ({}^{x}b)^{\varphi}], \\ \forall a, b \in \mathcal{G}, \quad \forall x \in \mathcal{L}_G \textit{ relative to } \mathcal{S} \rangle.$$

Let G be any polycyclic group with polycyclic generating sequence $\mathfrak{G}$. Taking $\mathcal{L}_G$ to be $\mathfrak{G}$, we see that $\nu(G)$ is finitely presented.

Algorithm (B, Moravec, and Morse)

Given a finite presentation for the polycyclic group $G = \langle \mathcal{G} \mid \mathcal{R} \rangle$ with polycyclic generating sequence $\mathfrak{G}$, the nonabelian tensor square $G \otimes G$ is computed by the following procedure.

1. Construct a finite presentation of $\nu(G)$ from $\mathcal{G}$, $\mathcal{R}, \mathfrak{G}$.
2. Compute a polycyclic presentation for $\nu(G)$.
3. Return the subgroup $[G, G^{\varphi}]$ of $\nu(G)$ as a polycyclic group.

Theorem (B, Moravec, and Morse)

Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then

$$G \otimes G \cong \Gamma(G^{\mathsf{ab}}) \times G \wedge G,$$

where $\Gamma(G^{\mathsf{ab}})$ is the Whitehead quadratic functor.

Motivation for investigation: finite p-groups

Using the small groups library of GAP and the implemented algorithms of B, Moravec and Morse (2008), we calculated nonabelian tensor squares of numerous finite p-groups, with the initial goal of bounding the order of the nonabelian tensor squares. We noticed consistent patterns for odd primes:

$$|G \otimes G| = |G'| \cdot |H_2(G)| \cdot |\Gamma(G^{\mathsf{ab}})| = |G \wedge G| \cdot |\Gamma(G^{\mathsf{ab}})|.$$

The situation for $p = 2$ is less consistent.

Derived and Lower Central Series

Rocco (1991) describes the derived and lower central series of $\nu(G)$. Using identities in $\nu(G)$, we have

Proposition

Let G be any group. Then the following hold:

(i) *For every $n \geq 0$, $[G, G^{\varphi}]^{(n)} = [G^{(n)}, (G^{(n)})^{\varphi}]$.*

(ii) *For every $n \geq 1$,*
$\gamma_{n+1}([G, G^{\varphi}]) = [\gamma_n(G'), G'^{\varphi}] = [G', \gamma_n(G')^{\varphi}]$.

One therefore recovers:

Corollary

(i) If G is a solvable group of derived length d, then $G \otimes G$ is solvable of derived length at most d.
(ii) If G is a nilpotent group of class c, then $G \otimes G$ is nilpotent of class at most $\leq \lfloor \frac{c+1}{2} \rfloor$.

Structure of nonabelian tensor square

The following improves Proposition 3.3 of Rocco (1994).

Lemma

Let G be a group such that G^{ab} is finitely generated by $\{x_i G' | 1 \leq i \leq s\}$. Set $E(G)$ to be $\langle [x_i, x_j^{\varphi}] | i < j \rangle [G', G^{\varphi}]$. Then:

(i) $\nabla(G)$ *is generated by the elements of the set* $\{[x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] | 1 \leq i < j \leq s\}$.

(ii) $G \otimes G = \nabla(G) E(G)$.

Structure of nonabelian tensor square: Main Result

Consequently, generalizing Proposition 8 of Brown, Johnson and Robertson and Proposition 3.1 of B, Moravec, and Morse, we have:

Theorem

Assume that G^{ab} is finitely generated. Then the following hold:

(i) *The map f_1 defined to be the restriction $f|_{\nabla(G)} : \nabla(G) \to \nabla(G^{ab})$ of the projection onto G^{ab}, has kernel $N = E(G) \cap \nabla(G)$. Moreover, N is a central elementary abelian 2-subgroup of $G \otimes G$ of rank at most the 2-rank $rk_2(G^{ab})$ of G^{ab}.*

(ii) $(G \otimes G)/N \simeq \nabla(G^{ab}) \times (G \wedge G)$.

(iii) *Suppose either that G^{ab} has no elements of order two or that G' has a complement in G. Then $\nabla(G) \simeq \nabla(G^{ab})$ and $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.*

Structure of nonabelian tensor square

Corollary

Let G be a group such that G^{ab} is a finitely generated abelian group with no elements of order two. Then

$$J(G) \simeq \Gamma(G^{\mathsf{ab}}) \times H_2(G).$$

Theorem

Let G be a group and let F be a free group such that $G \simeq F/R$ for some normal subgroup R of F. Then

$$G \wedge G \simeq \frac{F'}{[F, R]}.$$

The proof is based on the proof of Theorem 2 of Miller (1952).

Structure of nonabelian tensor square

We recover known results:

Corollary (Brown, Johnson and Robertson, Proposition 6)

Let F_n be a free group of rank n. Then

$$F_n \otimes F_n \simeq \mathbb{Z}^{n(n+1)/2} \times (F_n)'.$$

Corollary (B, Moravec, and Morse, Corollary 1.7)

Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of rank $n > 1$ and class $c \geq 1$. Then

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times (\mathcal{N}_{n,c+1})'.$$

Structure of nonabelian tensor square: free solvable

Corollary

Let F be the free group of finite rank $n > 1$, and let $G = F/F^{(d)}$ be the free solvable group of derived length d and rank $n > 1$. Then

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times F'/[F, F^{(d)}]$$

is an extension of a nilpotent group of class ≤ 3 by a free solvable group of derived length $d - 2$ and infinite rank. In particular, if $d = 2$, then $G \otimes G$ is a nilpotent group.

Application to a particular p-group

F_d = free group on d generators. For every integer i the group $\gamma_i(F_d)/\gamma_{i+1}(F_d)$ is free abelian of rank

$$m_d(i) := \frac{1}{i} \sum_{t|i} \mu(t) d^{i/t}$$

(where μ is the Mobius function).

Lower central p-series $\{\lambda_i(G)_{i\geq 1}\}$ of G:

$$\lambda_1(G) = G$$
$$\lambda_{k+1}(G) = [\lambda_k(G), G]\lambda_k(G)^p, \text{ for any } k \geq 1.$$

Let $G_{d,c} = F_d/\lambda_{c+1}(F_d)$, a finite p-group of class c and order p^m, with $m = \sum_{j=1}^{c}(c+1-j)m_d(j)$.

Application to a particular p-group

Proposition

$$G_{d,c} \wedge G_{d,c} \simeq (G_{d,c+1})'$$

and

$$G_{d,c} \otimes G_{d,c} \simeq (\mathbb{Z}_{p^c})^{d(d+1)/2} \times (G_{d,c+1})'.$$

Non-abelian tensor square of finite p-groups

Lemma

Let G be a finite p-group, then for every $k \geq 1$,

$$[\lambda_k(G), G^{\varphi}] = [G, (\lambda_k(G))^{\varphi}].$$

The following result is an improvement of Rocco (1991), Corollary 3.12 and is proved also by A. McDermott in his PhD Thesis.

Proposition

Let G be a finite group of order p^n (p a prime) and let $d = d(G)$ be the minimum number of generators of G. Then

$$p^{d^2} \leq |[G, G^{\varphi}]| \leq p^{nd}.$$

The upper bound is best possible, e.g., $F_2/\lambda_3(F_2)$.

Bound on the order of the Schur multiplier of finite p-groups

Corollary

Let G be a finite p-group of order p^n with d generators.
If p is odd, then $|H_2(G)| \leq p^{d(n-(d+1)/2)}$.
If $p = 2$, then $|H_2(G)| \leq 2^{d(n-(d+3)/2)}$.