An Example of Counting Generators in Finite Groups

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Outline

1. Introduction
2. Motivation
3. Magma/GAP-friendly Tool
4. Results

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- $I_G \equiv$ Augmentation Ideal of G
- $pr(G) = d(G) - d(I_G)$ (the “presentation rank")

Theorems and Conjectures

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- Profinite Grushko-Neumann *Conjecture*: If H_1 and H_2 are profinite groups, then is $d(H_1 \amalg H_2) = d(H_1) + d(H_2)$?

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- False if G is solvable
- False if $pr(G) = 0$

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  3. and $G = \langle H_1, \ldots, H_s \rangle$ with $pr(G) = 0$
- Then $d(G) \leq r + s - 1$
- Also true if $pr(G) > 0$?

Main Conjecture

Special Case:
Conjecture: Let P be a p-group and Q be a q-group for distinct primes p, q, and suppose $d(P) \leq d$ and $d(Q) \leq d$. Then for all finite groups G such that $G = \langle P, Q \rangle$, G is $(d+1)$-generated.

$P(G, s)$ is born

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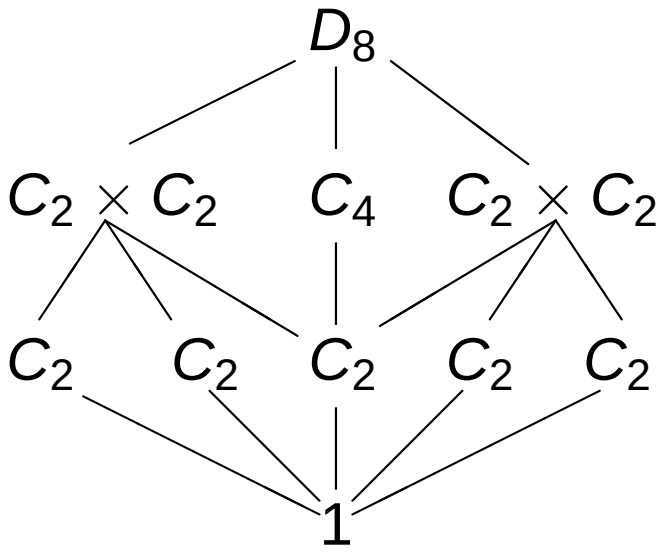
- Equivalently:

$$P(G, s) = \sum_{H \leq G} \frac{\mu(H)|H|^s}{|G|^s} = \sum_{H \leq G} \frac{\mu(H)}{|G : H|^s}$$

Subgroup Lattice for Dihedral Group of Order 8



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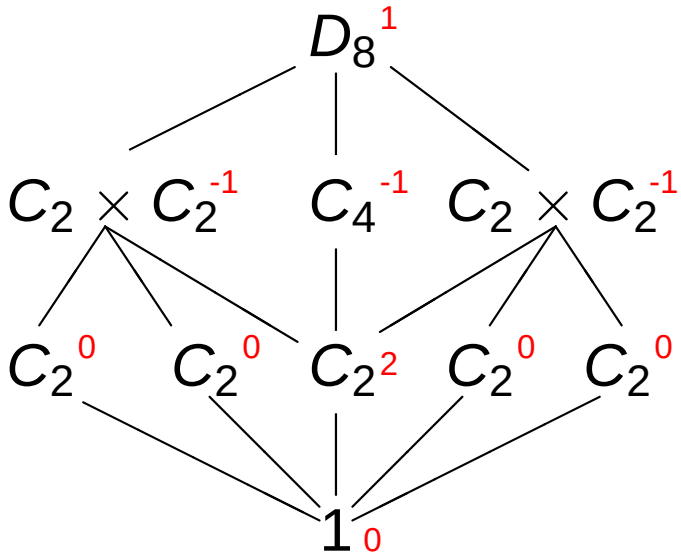

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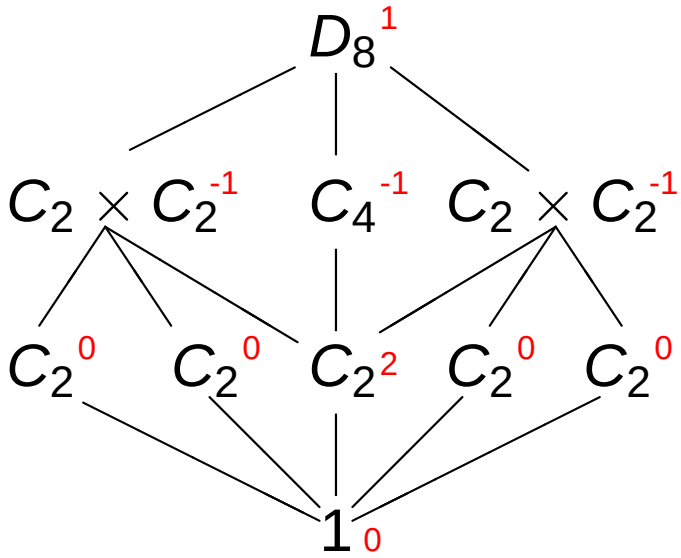

- So $P(D_8, s) = 1 + \frac{-1}{2^s} + \frac{-1}{2^s} + \frac{-1}{2^s} + \frac{0}{4^s} + \frac{0}{4^s} + \frac{2}{4^s} + \frac{0}{4^s} + \frac{0}{4^s} + \frac{0}{8^s}$

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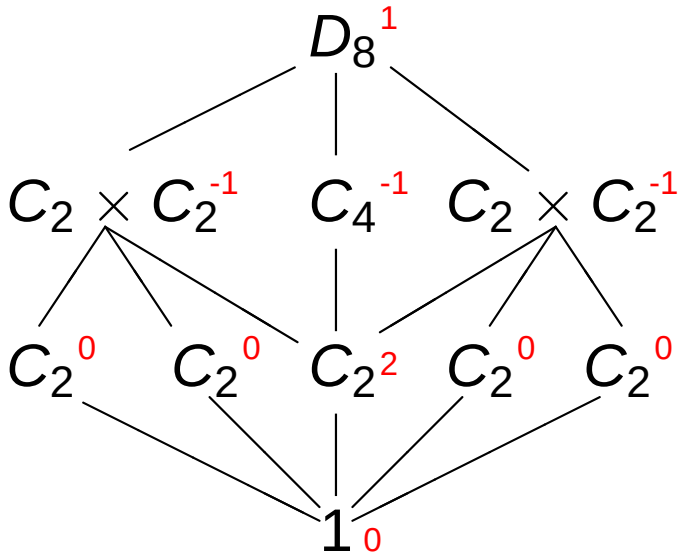

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- $P(D_8, s) = 1 - \frac{3}{2^s} + \frac{2}{4^s}$

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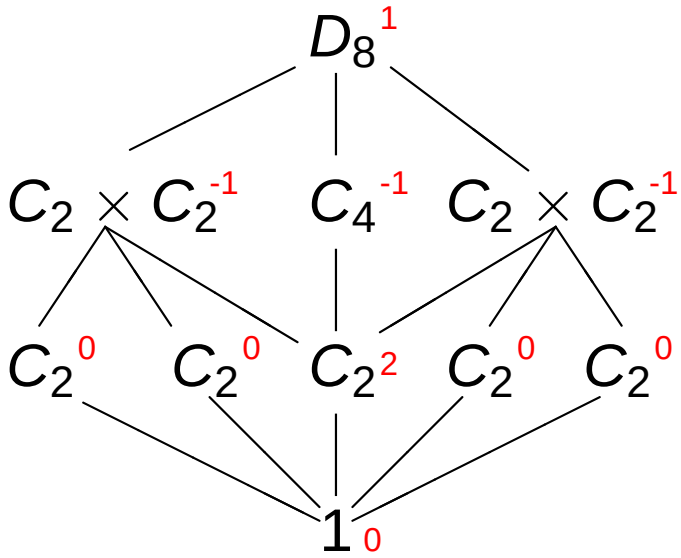

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- $P(D_8, s) = 1 - \frac{3}{2^s} + \frac{2}{4^s}$
- $P(D_8, s) = (1 - \frac{2}{2^s})(1 - \frac{1}{2^s})$

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- G solvable if and only if $P(G, s) = \prod(1 - \frac{c_i}{q_i^{n_i}})$
- $P(G, s) = \frac{\phi(G,s)}{|G|^s}$, so $\phi(G, s) = |G|^s P(G, s)$

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- $P(A_5, s) = 1 - \frac{5}{5^s} - \frac{6}{6^s} - \frac{10}{10^s} + \frac{20}{20^s} + \frac{60}{30^s} - \frac{60}{60^s}$

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- $P(S_5, s) = P(C_2, s)P(A_5, s)$

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Conjecture: Let P be a p-group and Q be a q-group for distinct primes p, q, and suppose $d(P) \leq d$ and $d(Q) \leq d$. Then for all finites groups G such that $G = \langle P, Q \rangle$, G is $(d+1)$-generated.

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- (Dalla Volta and Lucchini): Minimal counterexamples:

$$L_t = \{(l_1, \ldots, l_t) \in L^t \ : \ l_1 \equiv \cdots \equiv l_t \pmod{M}\}$$

for some group L with unique minimal normal subgroup M.

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- $M = S^n$, S simple nonabelian
- $L/M \cong L_t/M^t$, so L/M is $(d+1)$-generated

Main Theorem

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Main Theorem: Let $G \cong L_t$ be a minimal counterexample to this conjecture. Then either $d(L/M) = d + 1$ or L/M is nonsolvable.

Sketch of Proof

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- For $|G| \geq 120$, use theoretical bounds based on the factorization (G is solvable).
- For $|G| < 120$, use Magma to check all groups individually.

Sketch of Proof

Once we have those lemmas, we combine them to get:

- *Lemma:* Let $M = S^n$. Then $\frac{P(L/M,d+1)}{P(L/M,d)} > f(n,S,d)$.

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- *Main Theorem* Let $G \cong L_t$ be a minimal counterexample to this conjecture. Then either $d(L/M) = d+1$-generated or L/M is nonsolvable.
- *Sketch of Proof:* Suppose $d(L/M) \leq d$ and L/M solvable. Show that we contradict the lemma above.

Thank you!

Questions?

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