Computing homological functors of polycyclic groups

A brief history on computing the nonabelian tensor squares and exterior squares of polycyclic groups.

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Clair Miller 1952

Let G be any group.

$$0 \longrightarrow H_2(G) \longrightarrow G \wedge G \xrightarrow{\kappa'} G' \longrightarrow 1$$

Miller wanted to give a group theoretic interpretation of the second homology group with integral coefficients.

The wedge product captures the universal commutator relations so that the kernel $H_2(G)$ measures the extent that the commutator relations fail to be a consequence of the universal commutators relations, in particular $[x,x]=1$, $[x,y][y,x]=1$, ${}^{x}[y,z]=[x,[y,z]][y,z]$ and $[xy,z]={}^{x}[y,z][x,z]$.

The wedge product is the nonabelian exterior square of G.

R. Keith Dennis (1976 Unpublished)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_2(G) & \longrightarrow & G \otimes G & \xrightarrow{\kappa} & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H_2(G) & \longrightarrow & G \wedge G & \xrightarrow{\kappa'} & G' & \longrightarrow & 1
\end{array}$$

Dennis extends Miller's work in his preprint "In Search of New 'Homology' Functors Having a Close in Relation to K-Theory".

He was interested in the group we label $J_2(G)$ as a generalization of the homology groups that meets some K-theoretical conditions.

The tensor product is called the nonabelian tensor square.

Brown and Loday (1987)

Brown and Loday introduce the nonabelian tensor product of two groups that act compatibly on each other and by conjugation on themselves.

If we take the compatible actions to be conjugation we can define the nonabelian tensor square.

The group $J_2(G)$ is isomorphic to the third homotopy group of the suspension of an Eilenberg-MacLane space.

Brown and Loday (1987)

A covering group $\hat{G}$ of a group G is a central extension

$$1 \longrightarrow H_2(G) \stackrel{\iota}{\longrightarrow} \hat{G} \longrightarrow G \longrightarrow 1$$

such that image ι is contained in $\hat{G}'$.

Corollary 1. *Let G be a group. If $H_2(G)$ is finitely generated then $G \wedge G \cong \hat{G}'$.*

Polycyclic groups are finitely generated and hence have finitely generated Schur multipliers.

Let $G = F_n/R$ where F_n is the free group of rank n be a polycyclic group. The exterior square of G can be found by computing the derived subgroup the free central extension

$$F_n/[F_n, R]$$

which is a covering group of G.

Brown and Loday (cont.)

Exact rows and central extensions as columns.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \longrightarrow & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & \downarrow & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & 0 & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Variation of the diagram

Exact rows and central extensions as columns.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & \downarrow & & \kappa\downarrow & & \kappa'\downarrow & & \\
 & & 0 & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Possible configurations of the diagram

These do not need to hold always.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{ab}) & \longrightarrow & \nabla(G)\times H_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G\wedge G\times\nabla(G) & \longrightarrow & G'\times H_2(G) & \longrightarrow & 1 \\
 & & \downarrow & & \kappa\downarrow & & \kappa'\downarrow & & \\
 & & 0 & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Brown and Loday (cont.)

Given a group G we define the nonabelian tensor square $G \otimes G$ as the group generated by the symbols

$$g \otimes g' \quad \text{for all } g, g' \in G$$

subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$$
$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

where ${}^{x}y = xyx^{-1}$.

The presentation above exhibits the universal properties of the nonabelian tensor square, but it does not reflect its group structure very well.

Brown, Johnson and Robertson (1987)

The investigation of this group construction from a group theoretic view started with a paper by Ronald Brown, David Johnson, and Edmund Robertson (1987).

The goals of their investigation include:

- Compute the nonabelian tensor square for a given G. That is, give a description of $G \otimes G$ that is simplified and easy to recognize.
- Determine the structure of $G \otimes G$ from the structure of G.
- Compute homomorphisms of $G \otimes G$.

Structure Results

If G is finite then $G \otimes G$ is finite.

If G' is nilpotent of class c then $G \otimes G$ is nilpotent of class at most $c+1$.

If G' is solvable of derived length l then $G \otimes G$ is solvable with derived length at most $l+1$.

If G is polycyclic then $G \otimes G$ is polycyclic.

The nonabelian exterior square has these properties as well.

Computing the tensor square

For a finite group G, the definition gives us a finite presentation of $G \otimes G$. We can apply Tietze transformations to this presentation to obtain a simplified presentation of $G \otimes G$. We can then examine this simplified presentation to determine (in a more standard way) what the tensor square is.

Brown, Johnson, and Robertson (1987) compute the nonabelian tensor square of all nonabelian groups up to order 30 using Tietze transformations. The definition of the nonabelian tensor square gives $|G|^2$ generators and $2|G|^3$ relations.

This method does not scale well.

Infinite Groups

In the Brown, Johnson, and Robertson paper, they define the concept of a crossed pairing. Let G and L be groups. We call the mapping

$$\Phi : G \times G \longrightarrow L$$

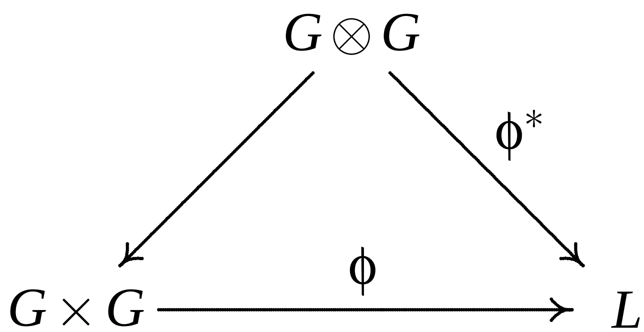
a crossed pairing if for all $g, g', g'' \in G$:

$$\Phi(gg', g'') = \Phi({}^{g}g, {}^{g}g'')\Phi(g, g'')$$
$$\Phi(g, g'g'') = \Phi(g, g')\Phi({}^{g'}g, {}^{g'}g'').$$

This mapping lifts to a homomorphism $\Phi^* : G \otimes G \longrightarrow L$ such that $\Phi^*(g \otimes g') = \Phi(g, g')$ for all $g, g' \in G$.

Infinite Groups (cont.)

We have the commutative diagram.



Crossed pairings give a method for computing the nonabelian tensor square of infinite groups G.

1. Conjecture a group L.

2. Construct a crossed pairing $\Phi : G \times G \to L$.

3. Show the Φ^* is an isomorphism.

Free Nilpotent groups of Class 2 of finite rank

The free nilpotent of class 2 case was settled by Bacon (1994):

Theorem 2. *Let* $\mathcal{N}_{n,2}$ *be a free nilpotent of class* 2 *group of rank n. Then* $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2}$ *is isomorphic to the free abelian group of rank* $n(n^2+3n-1)/3$.

This result was proved using crossed pairings.

Checking to see if one has a crossed pairing and that the lift is an isomorphism is relatively straightforward since the nonabelian tensor square is abelian.

2-Engel groups

The next logical step would be to compute the nonabelian tensor square of the free nilpotent groups of class 3.

A reasonable intermediate step was the class of 2-Engel groups i.e. groups that satisfy the law $[[x,y],y] = 1$. Free 2-Engel groups of rank greater than 2 are nilpotent of class 3.

The free 2-Engel group of rank 2 is the free nilpotent of class 2 group of rank 2.

Corollary 3. *The nonabelian tensor square of the free* 2*-Engel group of rank* 2 *is isomorphic to the free abelian group of rank* 6*.*

The free 2-Engel groups of rank $n > 2$ are metabelian, nilpotent of class 3 and $\gamma_3(G)$ has exponent 3. Every 2-generator subgroup is nilpotent of class 2.

Free 2-Engel groups of rank 3

Denote the free 2-Engel group of rank n by $\mathcal{E}(n,2)$.

The nonabelian tensor square of the free 2-Engel group of rank 3 turns out not to be abelian.

The Burnside groups of exponent 3 and rank n is a finite homomorphic image of the free 2-Engel group of rank n.

We found, using a program from Graham Ellis, that $\mathcal{B}(3,3) \otimes \mathcal{B}(3,3)$ was isomorphic to

$$N \times A$$

were A is an elementary abelian 3-group of rank 11 and N is a nilpotent of class 2 group of exponent 3.

Free 2-Engel groups of rank 3 (cont.)

The group $\mathcal{E}(3,2)$ has a polycyclic presentation hence every element has a unique collected form.

Let g and g' be arbitrary elements of $\mathcal{E}(3,2)$ generated by g_1, g_2, and g_3. Then

$$\begin{aligned} g &= g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} [g_1,g_2]^{\beta_{1,2}} [g_1,g_3]^{\beta_{1,3}} [g_2,g_3]^{\beta_{2,3}} [g_1,g_2,g_3]^{\gamma_{1,2,3}} \\ g' &= g_1^{\alpha'_1} g_2^{\alpha'_2} g_3^{\alpha'_3} [g_1,g_2]^{\beta'_{1,2}} [g_1,g_3]^{\beta'_{1,3}} [g_2,g_3]^{\beta'_{2,3}} [g_1,g_2,g_3]^{\gamma'_{1,2,3}} \end{aligned}$$

where the α_i, α'_i, $\beta_{i,j}$, $\beta'_{i,j}$ are integers and $\gamma_{i,j,k}$ and $\gamma'_{i,j,k}$ are integers modulo 3.

We define a multiplication and conjugation formulas (as Hall polynomials) for elements in this group.

Free 2-Engel groups of rank 3 (cont.)

For example:

$$g \cdot g' = g_1^{\alpha_1^*} g_2^{\alpha_2^*} g_3^{\alpha_3^*} [g_1, g_2]^{\beta_{1,2}^*} [g_1, g_3]^{\beta_{1,3}^*} [g_2, g_3]^{\beta_{2,3}^*} [g_1, g_2, g_3]^{\gamma_{1,2,3}^*}$$

where

$$\begin{aligned}
\alpha_1^* &= \alpha_1 \alpha_1', \quad \alpha_2^* = \alpha_2 \alpha_2', \quad \alpha_3^* = \alpha_3 \alpha_3' \\
\beta_{1,2}^* &= \beta_{1,2} + \beta_{1,2}' - \alpha_1 \alpha_2', \quad \beta_{1,3}^* = \beta_{1,3} + \beta_{1,3}' - \alpha_1 \alpha_3', \\
\beta_{2,3}^* &= \beta_{2,3} + \beta_{2,3}' - \alpha_2 \alpha_3', \quad \text{and} \\
\gamma_{1,2,3}^* &= \gamma_{1,2,3} + \gamma_{1,2,3}' + \beta_{2,3} \alpha_1' + \beta_{1,2} \alpha_3' - \alpha_1' \alpha_2 \alpha_3 \\
&\quad + \alpha_1' \alpha_2' \alpha_3 - \alpha_1' \alpha_2 \alpha_3' \mod 3
\end{aligned}$$

Computing the nonabelian tensor square of $\mathcal{E}(3,2)$

An arbitrary generator of $\mathcal{E}(3,2)\otimes\mathcal{E}(3,2)$ is $g\otimes g'$ can be represented as

$$g_1^{\alpha_1}\cdot g_2^{\alpha_2}\cdot g_3^{\alpha_3}\cdot[g_1,g_2]^{\beta_{1,2}}\cdot[g_1,g_3]^{\beta_{1,3}}\cdot[g_2,g_3]^{\beta_{2,3}}\cdot[g_1,g_2,g_3]^{\gamma_{1,2,3}}\otimes$$
$$g_1^{\alpha'_1}\cdot g_2^{\alpha'_2}\cdot g_3^{\alpha'_3}\cdot[g_1,g_2]^{\beta'_{1,2}}\cdot[g_1,g_3]^{\beta'_{1,3}}\cdot[g_2,g_3]^{\beta'_{2,3}}\cdot[g_1,g_2,g_3]^{\gamma'_{1,2,3}}$$

where the α_i, α'_i, $\beta_{i,j}$, $\beta'_{i,j}$ are integers and $\gamma_{i,j,k}$ and $\gamma'_{i,j,k}$ are integers modulo 3.

We use the relations of the tensor square to expand this arbitrary generator (similar to commutator expansion) into a product of elements in $\mathcal{E}(3,2)\otimes\mathcal{E}(3,2)$.

$$xy\otimes z=({}^{x}y\otimes{}^{x}z)(x\otimes z)$$
$$x\otimes yz=(x\otimes y)({}^{y}x\otimes{}^{y}z)$$

Computing the nonabelian tensor square $\mathcal{E}(3,2)$(cont)

Each term of this expansion product has one of the following forms:

$$g_i^{\alpha_i} \otimes g_j^{\alpha_j}$$

$$g_i^{\alpha_i} \otimes [g_j, g_k]^{\beta_{j,k}}, \quad g_i^{\alpha_i} \otimes [g_j^{\alpha_j}, g_k^{\alpha_k}],$$

$$[g_j^{\alpha_j}, g_k^{\alpha_k}] \otimes g_i^{\alpha_i}, \quad [g_j, g_k]^{\beta_{j,k}} \otimes g_i^{\alpha_i}$$

$$g_i^{\alpha_i} \otimes [g_j, g_k, g_l]^{\gamma_{j,k,l}}, \quad g_i^{\alpha_i} \otimes [[g_j, g_k]^{\beta_{j,k}}, g_l^{\alpha_l}],$$

$$[g_j, g_k, g_l]^{\gamma_{j,k,l}} \otimes g_i^{\alpha_i} \quad [[g_j, g_k]^{\beta_{j,k}}, g_l^{\alpha_l}] \otimes g_i^{\alpha_i}$$

$$[g_i, g_j]^{\beta_{i,j}} \otimes [g_k, g_l]^{\beta_{k,l}}$$

$$[g_i, g_j]^{\beta_{i,j}} \otimes [g_j, g_k, g_l]^{\gamma_{j,k,l}}, \quad [g_j, g_k, g_l]^{\gamma_{j,k,l}} \otimes [g_i, g_j]^{\beta_{i,j}}$$

This expansion product has hundreds of terms in it and was done symbolically using GAP. This is possible since n is fixed at 3.

We then collected these terms which gives Hall polynomials for exponents of each term.

Computing the nonabelian tensor square $\mathcal{E}(3,2)$ (cont.)

Example of one of the exponents:

$$\begin{aligned}\tau_{1,2,1,3} = & -\alpha_1'\gamma_{1,2,3} + \alpha_1\gamma_{1,2,3}' + \alpha_1\alpha_1'\beta_{2,3} - \alpha_1\alpha_2'\beta_{1,3} + \alpha_1\alpha_3'\beta_{1,2} \\ & -\alpha_1\alpha_1'\beta_{2,3}' + \alpha_1'\alpha_2\beta_{1,3}' - \alpha_1'\alpha_3\beta_{1,2}' + \beta_{1,2}\beta_{1,3}' - \beta_{1,2}'\beta_{1,3} \\ & +\beta_{2,3}\binom{\alpha_1'}{2} - \alpha_1'\alpha_2'\beta_{1,3} + \alpha_1'\alpha_3'\beta_{1,2} + \alpha_1\alpha_2\beta_{1,3}' - \alpha_1\alpha_2\beta_{1,2}' \\ & -\beta_{2,3}'\binom{\alpha_1}{2} - \alpha_1\alpha_1'\alpha_2\alpha_3 + \alpha_1\alpha_1'\alpha_2'\alpha_3' - \alpha_1\alpha_1'\alpha_2\alpha_3' \\ & +\alpha_1\alpha_1'\alpha_2'\alpha_3 + \alpha_2'\alpha_3\binom{\alpha_1}{2} - \alpha_2\alpha_3'\binom{\alpha_1'}{2} - \alpha_2\alpha_3'\binom{\alpha_1}{2} \\ & +\alpha_2'\alpha_3\binom{\alpha_1'}{2} - \alpha_3'\beta_{1,2} + \alpha_2\beta_{1,2}'.\end{aligned}$$

These polynomials were determined having GAP implement the expansion and collection rules. Which was possible as $n = 3$.

Computing the nonabelian tensor square $\mathcal{E}(3,2)$(cont.)

We devise an $L = N \times A$ following our pattern from the $B(3,3)$.

Let F_6 be the free group of rank 6 and set $\mathcal{N}_{3,2}$ to be $F_n/\gamma_3(F_n) = \langle y_{1,2}, y_{2,1}, y_{1,3}, y_{3,1}, y_{2,3}, y_{3,2}\rangle$ the free nilpotent group of class two and rank 6. Set

$$\begin{aligned} R = \langle & [y_{1,2}, y_{2,1}], [y_{1,3}, y_{3,1}], [y_{2,3}, y_{3,2}], \\ & [y_{1,2}, y_{1,3}][y_{1,2}, y_{3,1}], [y_{1,2}, y_{2,3}][y_{1,2}, y_{3,2}], [y_{1,3}, y_{2,3}][y_{1,3}, y_{3,2}], \\ & [y_{1,2}, y_{1,3}][y_{2,1}, y_{1,3}], [y_{1,2}, y_{2,3}][y_{2,1}, y_{2,3}], [y_{1,2}, y_{2,1}][y_{2,1}, y_{2,1}], \\ & [y_{1,2}, y_{2,1}][y_{2,1}, y_{1,2}], [y_{1,3}, y_{3,1}][y_{3,1}, y_{1,3}], [y_{2,3}, y_{3,2}][y_{3,2}, y_{2,3}], \\ & [y_{1,2}, y_{3,1}]^3, [y_{1,2}, y_{2,3}]^3, [y_{1,3}, y_{2,3}]^3\rangle. \end{aligned}$$

These relations reflect the relations in the tensor square such as $[g_1, g_2] \otimes [g_2, g_3]^3 = 1_{\otimes}$ and $[g_1, g_2] \otimes [g_3, g_4] = ([g_1, g_2] \otimes [g_4, g_3])^{-1}$.

Set $N = \mathcal{N}_{3,2}/R$ and $A = F_{11}^{ab}$.

Computing the nonabelian tensor square $\mathcal{E}(3,2)$ (cont.)

After showing we had a crossed pairing which is again as messy as what we have just done. We obtained the following result.

Theorem 4 (BKM 1997)**.** *The nonabelian tensor square of the free 2-Engel group of rank* 3 *is a direct product of a free abelian group of rank* 11 *and an* 6*-generated nilpotent group of class 2 whose derived subgroup has exponent 3.*

Computing the nonabelian tensor square $\mathcal{E}(3,2)$ (cont.)

It was hard to see the structure of the group L.

We were worried about computing the tensor square not the exterior square or any other of the homological functors such as $\nabla(G)$ and $H_2(G)$.

It turns out that L is isomorphic to $B(3,2) \wedge B(3,2) \times \nabla(G)$.

Computing the nonabelian tensor square of $\mathcal{E}(n,2)$

The rank n case was even messier. We followed the same procedure via expansion, collection, setting up a crossed pairing and wending our way to an answer – of course guided by calculations of $B(3,n) \otimes B(3,n)$ for $n = 4,5,6$.

Theorem 5. *The nonabelian tensor square of the free 2-Engel group of rank $n > 2$ is a direct product of a free abelian group of rank $\frac{n(n^2+2)}{3}$ and an $n(n-1)$-generated nilpotent group of class 2 whose derived subgroup has exponent 3.*

All the details of this calculation is recored in Joanne Redden's dissertation (117 pages). The published result Blyth, Morse, Redden is in the Proceedings of the Edinburgh Mathematical Society (2004) (20 pages).

We were exhausted and the free nilpotent of class 3 and rank n case looked out of reach.

The group $\nu(G)$

Rocco (1991) considers the following group. Let G and G^{φ} be isomorphic groups via $\varphi : g \mapsto g^{\varphi}$, for all $g \in G$. Then

$$\nu(G) = \langle G, G^{\varphi} \mid {}^{k}[g, h^{\varphi}] = [{}^{k}g, \left({}^{k}h\right)^{\varphi}] = {}^{k^{\varphi}}[g, h^{\varphi}], \forall g, h, k \in G \rangle,$$

Rocco investigates the structural aspects of $\nu(G)$ relative to G:

Theorem 6. *Let G be a group.*

(i) If G is finite then $\nu(G)$ is finite.

(ii) If G is a finite p-group then $\nu(G)$ is a finite p-group.

(iii) If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c+1$.

(iv) If G is solvable of derived length d then $\nu(G)$ is solvable of class at most $d+1$.

The group $\nu(G)$ (cont)

The group $\nu(G)$ actually has its roots in a more general setting of crossed modules and was investigated for it potential for computing tensor product and squares by Ellis in Ellis and Leonard (1995).

The importance of this construction is the following:

Theorem 7. *Let G be a group. The map* $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ *defined by* $\sigma(g \otimes h) = [g, h^{\varphi}]$ *is an isomorphism.*

Ellis and Leonard (1995) give one more structure result.

Theorem 8. *Let G be a group. Then*

$$1 \longrightarrow [G, G^{\varphi}] \longrightarrow \nu(G) \longrightarrow G \times G \longrightarrow 1$$

is a short exact sequence.

The group $\nu(G)$ (cont)

Definition 9. Let G be a group and let $G = G_n \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1$ be a subnormal series for G. Let $\mathcal{T}_i$ denote a transversal for G_{i-1} in G_i and let $\mathcal{G}_i$ denote a lift of a generating set for G_i/G_{i-1} to $\mathcal{T}_i$. Set

$$\mathcal{L}_i = \begin{cases} \mathcal{G}_i, & \text{if } G_i/G_{i-1} \text{ is abelian} \\ \mathcal{T}_i, & \text{otherwise.} \end{cases}$$

Then define the set $\mathcal{L}_G$ to be

$$\mathcal{L}_G = \cup_{i=1}^{n} \mathcal{L}_i.$$

The group $\nu(G)$ (cont)

The following theorem is by A. McDermott (1998) which extends the work of Ellis:

Theorem 10. *Let G be a group generated by a set $\mathcal{G}$. Then $\nu(G) = G * G^{\varphi}/\langle J\rangle$, where J is the normal generating set consisting of the elements*

$$^{x}[a,b^{\varphi}][^{x}a,(^{x}b)^{\varphi}]^{-1}, {}^{x^{\varphi}}[a,b^{\varphi}][^{x}a,(^{x}b)^{\varphi}]^{-1}$$

for all a,b in $\mathcal{G}$ and x in $\mathcal{L}_G$.

Let $G = \langle \mathcal{G}, \mathcal{R}\rangle$. The theorem shows the defining presentation of $\nu(G)$ can be reduced to $2|\mathcal{G}|$ generators and $2|\mathcal{R}| + 2|\mathcal{G}|^2 \cdot |\mathcal{L}_G|$.

This a reasonably small presentation to compute and work with.

Two algorithms

For both algorithms we

1 Compute a finite presentation for $\nu(G)$ from some $\mathcal{L}_G$. For any finite group we can choose $G \trianglerighteq Z(G) \trianglerighteq 1$ to form $\mathcal{L}_G$.

Algorithm One:

2a Find a concrete representation for $\nu(G)$ (p-quotient algorithm, nilpotent quotient, solvable quotient, coset enumeration).

3a Compute $[G, G^{\varphi}]$.

Algorithm Two:

2b Compute the kernel of the mapping $\nu(G) \to G \times G$.

Polycyclic groups

Polycyclic groups are those groups with a finite subnormal series such that each factor is cyclic.

All polycyclic groups are solvable and have a well developed theory such that we can compute effectively with them.

Let G be a polycyclic group with generating set $\mathcal{G}$ and polycyclic generating set $\mathfrak{G}$. Then we can set $\mathcal{L}_G = \mathfrak{G}$. Moreover

Theorem 11. *Suppose that G is a polycyclic group. Then both $\nu(G)$ and $G \otimes G$ are also polycyclic.*

Polycyclic (cont.)

Corollary 12. *Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then the subgroup $[G, G^{\varphi}]$ of $\nu(G)$ is generated by $\{[\mathfrak{g}_i, \mathfrak{g}_i^{\varphi}], [\mathfrak{g}_i^{\varepsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}], [\mathfrak{g}_i, \mathfrak{g}_j^{\varphi}][\mathfrak{g}_j, \mathfrak{g}_i^{\varphi}]]\}$ for $1 \leq i < j, \leq k$,where*

$$\varepsilon = \begin{cases} 1, & \textit{if } |\mathfrak{g}_i| < \infty \\ \pm 1 & \textit{if } |\mathfrak{g}_i| = \infty \end{cases} \qquad \delta = \begin{cases} 1, & \textit{if } |\mathfrak{g}_j^{\varphi}| < \infty \\ \pm 1 & \textit{if } |\mathfrak{g}_j^{\varphi}| = \infty. \end{cases}$$

Observations

All “tensor” computations can be done as commutator calculations.

Lemma 13. *The following relations hold in $\nu(G)$:*

(i) ${}^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *and*
${}^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *for all* $g_1,g_{2,},g_3,g_4 \in G$;

(ii) $[g_1^{\varphi},g_2,g_3] = [g_1,g_2,g_3^{\varphi}] = [g_1^{\varphi},g_2,g_3^{\varphi}] = [g_1,g_2^{\varphi},g_3] = [g_1^{\varphi},g_2^{\varphi},g_3] =$
$[g_1,g_2^{\varphi},g_3^{\varphi}]$ *for all* $g_1,g_{2,},g_3 \in G$;

(iii) $[g,g^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g \in G$;

(iv) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g_1,g_2 \in G$;

(v) $[g,g^{\varphi}] = 1$ *for all* $g \in G'$.

We have nilpotency and solvability bounds on $\nu(G)$ which can make these computations easier when G is nilpotent or solvable.

Application

Using this theoretical framework, we compute the nonabelian tensor square of the free nilpotent groups of class 3 and finite rank.

All calculations involve working with the commutators in $[G, G^{\varphi}]$.

The problem involves finding exact structure of $[G, G^{\varphi}]$.

Using crossed pairings, the “simple” free 2-Engel case of rank *n* took 2 published papers totaling 33 pages plus an 117 page dissertation + 42 pages of supporting materials.

The free nilpotent class 3 rank *n* will be written up in about 7 published pages which includes all details.

Details have been written up for the J. of Algebra paper with Russell Blyth.

Free nilpotent groups of class 3

Theorem 14. *Let G be a free nilpotent group of class 3 and rank n. Then $G \otimes G \cong N \times A$ where N is nilpotent of class 2 with rank $n(n-1)$ and A is free abelian of rank $f(n)$ where*

$$\begin{aligned} f(n) &= n + 2\binom{n}{3} + 2\binom{n}{2} + 6\binom{n}{3} + 3\binom{n}{4} + 3\binom{n}{2} \\ &= n + 5\binom{n}{2} + 8\binom{n}{3} + 3\binom{n}{4} \\ &= \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}. \end{aligned}$$

But even this was tedious and difficult using basic commutators.

The free nilpotent groups of class c and rank n

To tackle the arbitrary class we needed structure theorems for a guide.

Theorem 15. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \Gamma(G^{ab}) \times G \wedge G.$$

The Whitehead quadratic functor $\Gamma(G^{ab})$ for a free abelian group of rank n is $F^{\mathrm{ab}}_{\binom{n+1}{2}}$.

Now the covering group of $\mathcal{N}_{n,c}$ is $\mathcal{N}_{n,c+1}$. By the corollary that $\mathcal{N}_{n,c} \wedge \mathcal{N}_{n,c}$ is isomorphic to the derived subgroup of $\mathcal{N}_{n,c+1}$.

Corollary 16. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \mathcal{N}'_{n,c+1} \times F^{\mathrm{ab}}_{\binom{n+1}{2}}.$$

A general observation

For the free nilpotent groups $J_2(G)$ is a direct product:

$$\Gamma(G^{ab}) \times H_2(G).$$

The reason for this is that G/G' is free abelian.

Corollary 17. *Let G be a group such that G/G' is free abelian. Then $J_2(G) \cong \nabla(G) \times H_2(G)$.*

Open question if $J_2(G)$ splits like this exactly when $G \otimes G \cong G \wedge G \times \nabla(G)$.

The Derived Subgroups of $\mathcal{N}_{n,c}$

Let $\mathcal{D}_{n,c}$ be the derived subgroup of $\mathcal{N}_{n,c}$ and $\mathcal{S}_{n,c}$ the set of simple basic commutators on n variables of length up to c.

Theorem 18. *Let $\mathcal{N}_{n,c}$ be the free nilpotent group of class $c \geq 1$ and rank $n \geq 1$. If $n = 1$ or $c = 1$ then $\mathcal{N}_{n,c}$ is abelian and $\mathcal{D}_{n,c}$ is trivial. If $n > 1$ and $c = 2$ then $\mathcal{D}_{n,c}$ is free abelian of rank $M(n,2) = \binom{n}{2}$. If $n > 1$ and $c > 2$ then*

$$\mathcal{D}_{n,c} \cong N_n \times F_f^{\mathrm{ab}},$$

where $f = |\mathcal{S}_{n,c} \setminus \mathcal{S}_{n,c-2}|$ and N_n is nilpotent of class $\lfloor c/2 \rfloor$.

The nonabelian tensor square of the free nilpotent group of class c and rank n

Using a formula counting the number of simple basic commutators shown by Gaglione and Spellman we obtain our final result.

Theorem 19. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 2$. Then*

$$G \otimes G \cong N_n \times F_g^{\mathrm{ab}},$$

where

$$g = \binom{n+c-1}{c+1}\left(c + \frac{(c+1)(c-1)}{n+c-1}\right) + \binom{n+1}{2}$$

and $N_{n,}$ is nilpotent of class $\lfloor c/2 \rfloor$ and has a minimal cardinality generating set with

$$\frac{(n+c-2)!((c+1)n-3n-c+3)+(c-1)!n!}{(c-1)!n!}$$

generators.

Other polycyclic groups

Let G be a torsion free crystallographic group. The group G is called a Bieberbach group. It is a group extension of a finite group P called the point group by a free abelian group L of finite rank called the lattice or translation group.

Hence we have the following short exact sequence.

$$1 \longrightarrow L \longrightarrow G \longrightarrow P \longrightarrow 1$$

The rank of L is called the dimension of G.

If G is the Bieberbach groups with point group the dihedral group of order 8 and dimension 4. Then

$$G \otimes G \cong G \wedge G \times \nabla(G) \cong G' \times H_2(G) \times \nabla(G)$$

where G' is a Bieberbach group with point group $\mathbb{Z}_2$ of dimension 3.

Other polycyclic groups (cont.)

The two generator p-groups that are nilpotent of class exactly 2.

Let G be such a group then it has a polycyclic generating sequence

$$a, b, [a, b].$$

The generators for $G \otimes G$ are

$$[a, a^{\varphi}],\ [b, b^{\varphi}],\ [a, b^{\varphi}],\ [a, b^{\varphi}][b, a^{\varphi}],\ [a, [a, b]^{\varphi}],\ [b, [a, b]^{\varphi}].$$

These generators form an abelian group. These generators are independent except in the case when $a = [a, b]$ or $b = [a, b]$ in this case we have $[a, a^{\varphi}] = [a, [a, b]^{\varphi}]$ or $[b, b^{\varphi}] = [b, [a, b]^{\varphi}]$.